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Contour Method for Resistive Evolution of Grad-Shafranov Plasma Equilibrium and Its Application to MTF studies at General Fusion

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INTRODUCTION

- General Fusion (GF) implements a Magnetized Target Fusion (MTF) approach in which a metal liner compresses a compact toroid plasma to fusion conditions.
- To guide the design of GF MTF systems, we need to simulate plasma dynamics inside a domain with moving boundaries. For this purpose, GF has developed the ISM-plasma code as a part of the Integrated System Model (ISM) project.
- The ISM-plasma code is based on assumption that slow resistive evolution of magnetically-confined plasma can be represented as a sequence of 2D axisymmetric Grad-Shafranov equilibria, which are linked through 1D transport processes [1].
- The code uses a mixed Eulerian-Lagrangian contour method: the Grad-Shafranov equilibrium is solved on a fixed mesh (Eulerian description) and transport processes are discretized on a set of contours advected by plasma motion (Lagrangian description).
- The main advantages of the ISM-plasma code are:
  - speed (the numerical time step is limited by a slow resistive diffusion timescale and not by a fast Alfven timescale as in usual MHD codes);
  - compatibility with moving boundaries of the plasma domain (especially useful for modeling MTF compression systems).
- We present several test cases to verify the ISM-plasma code against other MHD codes and demonstrate its capabilities.

CONTOUR METHOD

Assumptions

- System is axisymmetric, magnetic field in plasma is:

$$\mathbf{B} = \nabla\psi\times\nabla\phi + F\nabla\phi$$

- At any time t, plasma is in Grad-Shafranov equilibrium (instantaneous force balance):

$$\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \nabla p = 0 \quad\Rightarrow\quad \Delta^*\psi + FF'_\psi + \mu_0 r^2 p'_\psi = 0$$

- Toroidal field function $F(t,\psi)$ and plasma pressure $p(t,\psi)$ are functions of poloidal flux ψ.
- Dynamics of magnetic field, plasma density and pressure follow from single-fluid MHD:

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B} - \eta\nabla\times\mathbf{B}) \quad\Rightarrow\quad \begin{cases} \dfrac{\partial\psi}{\partial t} = -\mathbf{v}\cdot\nabla\psi + \eta\Delta^*\psi \\ \dfrac{\partial F}{\partial t} = r^2\nabla\cdot\left(-\dfrac{F}{r^2}\mathbf{v} + \dfrac{v_\phi}{r}\mathbf{B} + \dfrac{\eta}{r^2}\nabla F\right) \end{cases}$$

$$\frac{\partial n}{\partial t} = -\nabla\cdot(n\mathbf{v})$$

$$\frac{\partial p}{\partial t} = -\mathbf{v}\cdot\nabla p - \frac{5}{3}p\nabla\cdot\mathbf{v} + \frac{2}{3}\left(q^T + q^\Omega\right)$$

$$p = nk_BT$$

- Thermal conduction and Ohmic heating are:

$$q^T = \nabla\cdot\left(\frac{3}{2}\chi_\perp nk_B\nabla T\right)$$

$$q^\Omega = \frac{\eta}{\mu_0}\left(\frac{(\Delta^*\psi)^2}{r^2} + \frac{(\nabla F)^2}{r^2}\right)$$

- Thermal conduction is fast along magnetic field lines (parallel thermal diffusivity $\chi_\parallel\to\infty$), therefore plasma temperature $T(t,\psi)$ and density $n(t,\psi)$ are functions of poloidal flux ψ.
- Plasma magnetic diffusivity (electric resistivity) η and perpendicular thermal diffusivity $\chi_\perp$ are functions of other parameters, we assume they are constant on ψ contours.
- Problem: there is no explicit equation for plasma velocity **v**, it has to be determined from conditions that F, p, T and n are functions of ψ at any time; this is not straightforward!
- Solution: equations in ISM-plasma code are rewritten for passive scalars – quantities advected by plasma flow – poloidal flux ψ, toroidal flux Φ and total number of particles N inside ψ contour, and plasma entropy s; this removes the need to know **v**:

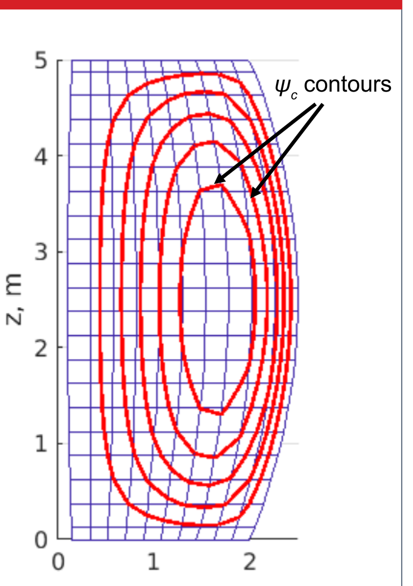
$$\Phi(t,\psi) = \int\limits_{\psi(r,z)>\psi} \frac{F(t,\psi(r,z))}{r}drdz$$

$$N(t,\psi) = \int\limits_{\psi(r,z)>\psi} n(t,\psi(r,z))\,2\pi r drdz$$

$$s(t,\psi) = p(t,\psi)n(t,\psi)^{-5/3}$$

Discretization

- 2D Eulerian mesh is introduced on r-z plane (indices j-k):
  - quadrilateral, structured, conformal to the domain;
  - z mesh coordinates (z_k, horizontal lines) are fixed;
  - r mesh coordinates ($r_{jk}=r_j(z_k)$, vertical lines) change in time if the right boundary (liner surface) is moving.
- Poloidal flux is discretized on this mesh, $\psi_{jk}=\psi(r_{jk},z_k)$.
- 1D functions of ψ (F, p, T, n) are discretized on an array of values $\psi_c=[\psi_0(\text{axis}), \psi_1, \psi_2, \dots \psi_C(\text{edge})]$.
- Levels $\psi(r,z)=\psi_c$ correspond to a set of contours, these contours form a Lagrangian mesh:
  - they follow the motion of plasma;
  - total number of plasma particles $N_c=N(t,\psi_c)$ inside each contour ψ_c remains constant in time;
  - values of ψ_c change during resistive dynamics.
- Every time step contains two stages:
  - finding new 2D Grad-Shafranov equilibrium;
  - advancing 1D contour-averaged transport equations.



First stage: 2D Grad-Shafranov equilibrium

- Grad-Shafranov Equation (GSE) is discretized on 2D Eulerian r-z mesh:

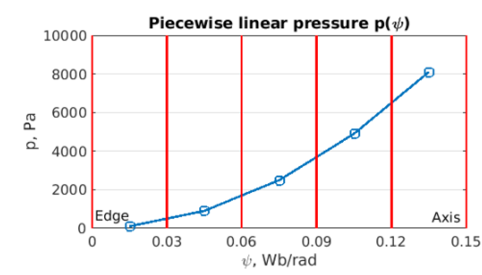
$$[\Delta^*\psi + FF'_\psi + \mu_0 r^2 p'_\psi]_{jk} = 0$$

- At t=0, functions $F(\psi)$ and $p(\psi)$ are given and GSE is solved for 2D poloidal flux ψ_{jk}.
- At t>0, functions $F(t,\psi)$ and $p(t,\psi)$ are reconstructed from current profiles of toroidal flux $\Phi(t,\psi)$, plasma entropy $s(t,\psi)$ and density $n(t,\psi)$:

$$F_{c+\frac{1}{2}} = \left.\frac{d\Phi(t,\psi)}{dL(t,\psi)}\right|_{c+\frac{1}{2}} \approx \frac{\Phi_{c+1}-\Phi_c}{L_{c+1}-L_c}$$

$$n_{c+\frac{1}{2}} = \left.\frac{dN(t,\psi)}{dV(t,\psi)}\right|_{c+\frac{1}{2}} \approx \frac{N_{c+1}-N_c}{V_{c+1}-V_c}$$

$$p_{c+\frac{1}{2}} = s_{c+\frac{1}{2}}n_{c+\frac{1}{2}}^{5/3}$$



- Inductance and volume inside contour ψ_c:

$$L_c = \int\limits_{\psi(r,z)>\psi_c}\frac{drdz}{r} \approx \sum_{\psi_{jk}>\psi_c}\frac{\Delta r_{jk}\Delta z_k}{r_{jk}}$$

$$V_c = \int\limits_{\psi(r,z)>\psi_c} 2\pi r drdz \approx \sum_{\psi_{jk}>\psi_c} 2\pi r_{jk}\Delta r_{jk}\Delta z_k$$

- At the first stage of every time step, GSE is solved for ψ_{jk} by iterations, keeping fixed values of ψ_c, Φ_c, $s_{c+1/2}$, N_c and updating L_c and V_c at each iteration based on new ψ_{jk}.

Second stage: 1D transport equations

- At the second stage of every time step, contour-averaged 1D transport equations for poloidal flux ψ_c, toroidal flux Φ_c and plasma entropy $s_{c+1/2}$ are advanced in time:

$$\frac{d\psi_c}{dt} = \eta_c\langle\Delta^*\psi\rangle_c$$

$$\frac{d\Phi_c}{dt} = Q_c^\Phi$$

$$\frac{ds_{c+\frac{1}{2}}}{dt} = \frac{2}{3}n_{c+\frac{1}{2}}^{-5/3}\left(q^T_{c+\frac{1}{2}} + q^\Omega_{c+\frac{1}{2}}\right)$$

- The contour-averaged terms on the right hand side and their discretization:

$$\langle\Delta^*\psi\rangle_c = \left.\frac{dI(t,\psi)}{dL(t,\psi)}\right|_c \approx \frac{I_{c+\frac{1}{2}} - I_{c-\frac{1}{2}}}{L_{c+\frac{1}{2}} - L_{c-\frac{1}{2}}}$$

$$Q_c^\Phi = \frac{1}{2\pi}\int\limits_c \frac{\eta}{r^2}\nabla F\cdot d\mathbf{S} = \eta_c I_c F'_\psi\big|_c \approx \eta_c I_c\frac{F_{c+\frac{1}{2}} - F_{c-\frac{1}{2}}}{\psi_{c+\frac{1}{2}} - \psi_{c-\frac{1}{2}}}$$

$$q^T_{c+\frac{1}{2}} + q^\Omega_{c+\frac{1}{2}} = \left.\frac{d(Q^T+Q^\Omega)}{dV}\right|_{c+\frac{1}{2}} \approx \frac{(Q^T_{c+1} - Q^T_c) + \Delta Q^\Omega_{c+\frac{1}{2}}}{V_{c+1}-V_c}$$

$$Q_c^T = \int\limits_c \frac{3}{2}\chi_\perp nk_B\nabla T\cdot d\mathbf{S} = \frac{3}{2}\chi_{\perp c}n_cK_ck_BT'_\psi\big|_c \approx \frac{3}{2}\chi_{\perp c}n_cK_ck_B\frac{T_{c+\frac{1}{2}} - T_{c-\frac{1}{2}}}{\psi_{c+\frac{1}{2}} - \psi_{c-\frac{1}{2}}}$$

$$\Delta Q^\Omega_{c+\frac{1}{2}} \approx \frac{\pi}{\mu_0}\left(\eta_c\langle\Delta^*\psi\rangle_c^2 + \eta_{c+1}\langle\Delta^*\psi\rangle_{c+1}^2\right)(L_{c+1}-L_c) + \frac{\pi}{\mu_0}\left(\eta_cI_cF'^2_\psi\big|_c + \eta_{c+1}I_{c+1}F'^2_\psi\big|_{c+1}\right)(\psi_{c+1}-\psi_c)$$

- Additional integrals ($-I_c/\mu_0$ gives toroidal plasma current inside contour ψ_c):

$$I_c = \frac{1}{2\pi}\int\limits_c\frac{1}{r^2}\nabla\psi\cdot d\mathbf{S} = \int\limits_{\psi(r,z)>\psi_c}\frac{\Delta^*\psi}{r}drdz \approx \sum_{\psi_{jk}>\psi_c}\frac{[\Delta^*\psi]_{jk}}{r_{jk}}\Delta r_{jk}\Delta z_k$$

$$K_c = \int\limits_c\nabla\psi\cdot d\mathbf{S} = \int\limits_{\psi(r,z)>\psi_c}\Delta\psi\,2\pi r drdz \approx \sum_{\psi_{jk}>\psi_c}[\Delta\psi]_{jk}\,2\pi r_{jk}\Delta r_{jk}\Delta z_k$$

Numerical algorithm

1. Solve GSE at t=0: find ψ_{jk} by iterations with given $F(\psi)$, $p(\psi)$. Choose contours ψ_c, determine contour values Φ_c, $s_{c+1/2}$, N_c
2. If domain is changing: move boundary, update r-z mesh.
3. Solve GSE: find ψ_{jk} by iterations, keeping fixed ψ_c, Φ_c, $s_{c+1/2}$, N_c and updating L_c and V_c at each iteration.
4. Advance 1D contour-averaged transport equations: find ψ_c, Φ_c, $s_{c+1/2}$ at next time step
5. $t=t_{end}$? No → return to step 2; Yes → Results

STEADY GEOMETRY

- We perform comparison between ISM-plasma and two other MHD codes, DELiTE [2] and OpenFOAM [3], for the test cases in steady geometry.

Test case 1: constant shaft current

- Plasma is formed in a domain with rectangular cross section and resistively decays; toroidal field at the boundary F_{edge} (shaft current) is kept constant in time.

ISM-plasma numerical parameters:

$$N_r = 200,\quad N_z = 60,\quad N_C = 20,\quad dt = 5\cdot10^{-6}\ \text{s}$$

Initial plasma parameters:

$$R_1 = 0.2\ \text{m},\quad R_2 = 1.2\ \text{m},\quad H = 0.6\ \text{m}$$

$$\bar\psi = \frac{\psi}{\psi_{axis}},\quad \psi_{axis} = 0.1\ \text{Wb/rad}$$

$$F(\bar\psi) = F_{edge} + (F_{axis} - F_{edge})\bar\psi^2$$

$$F_{edge} = 0.4\ \text{T·m},\quad F_{axis} = 0.74\ \text{T·m}$$

$$p(\bar\psi) = p_{edge} + (p_{axis} - p_{edge})\bar\psi$$

$$p_{edge} = 10^3\ \text{Pa},\quad p_{axis} = 10^4\ \text{Pa}$$

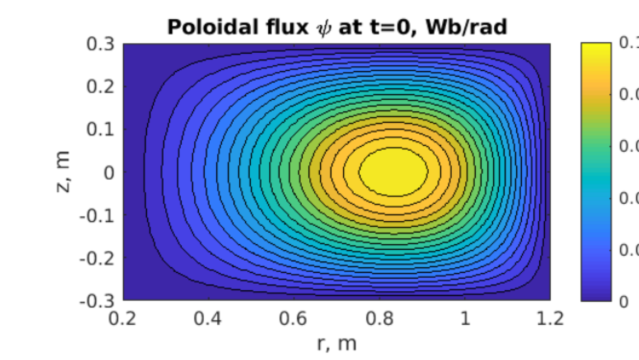
$$n(\bar\psi) = n_0\bar\psi,\quad n_0 = 7.24\cdot10^{20}\ \text{m}^{-3}$$

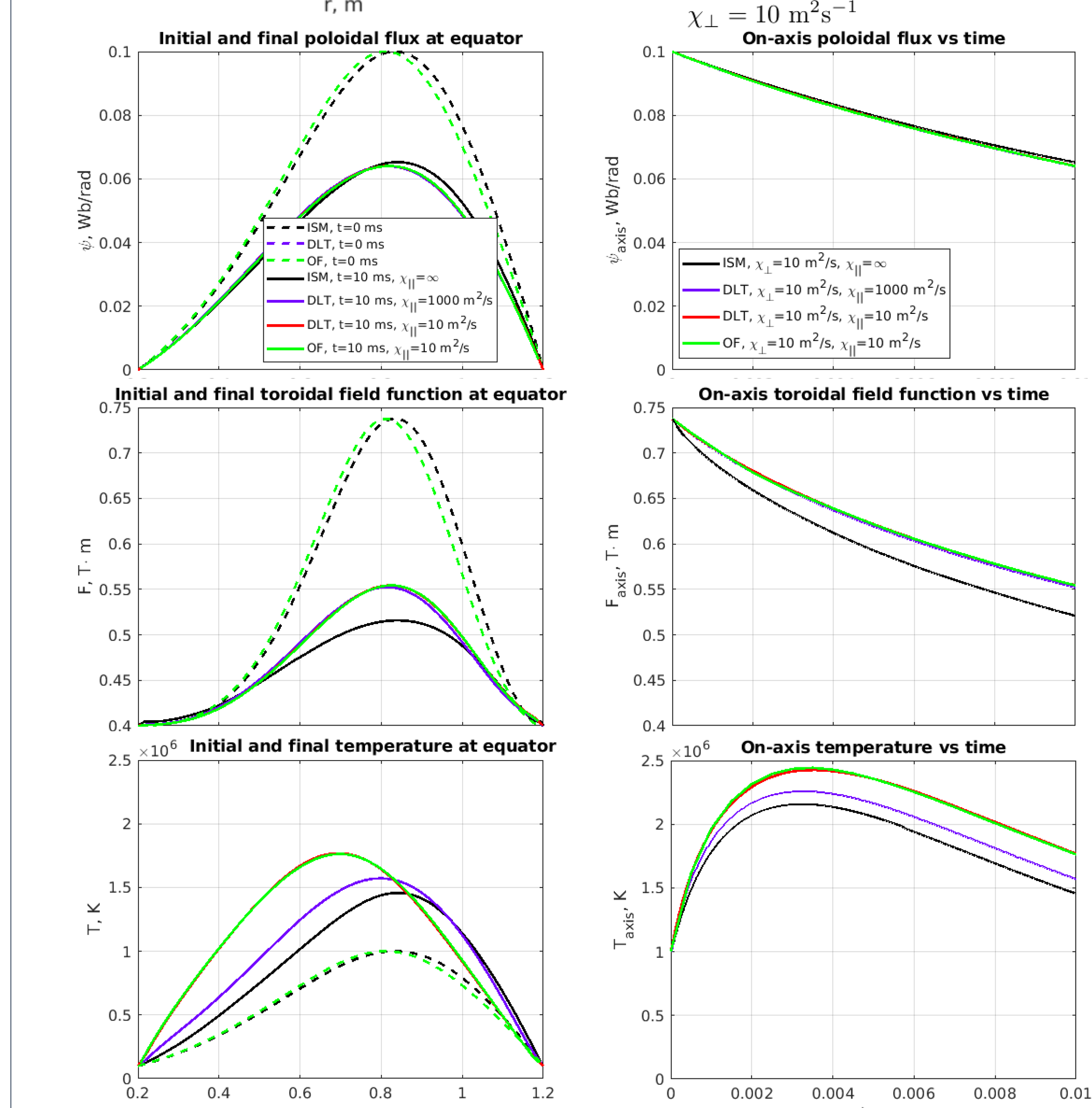
$$T(\bar\psi) = T_{edge} + (T_{axis} - T_{edge})\bar\psi$$

$$T_{edge} = 10^5\ \text{K},\quad T_{axis} = 10^6\ \text{K}$$

$$\eta = 1\ \text{m}^2\text{s}^{-1}$$

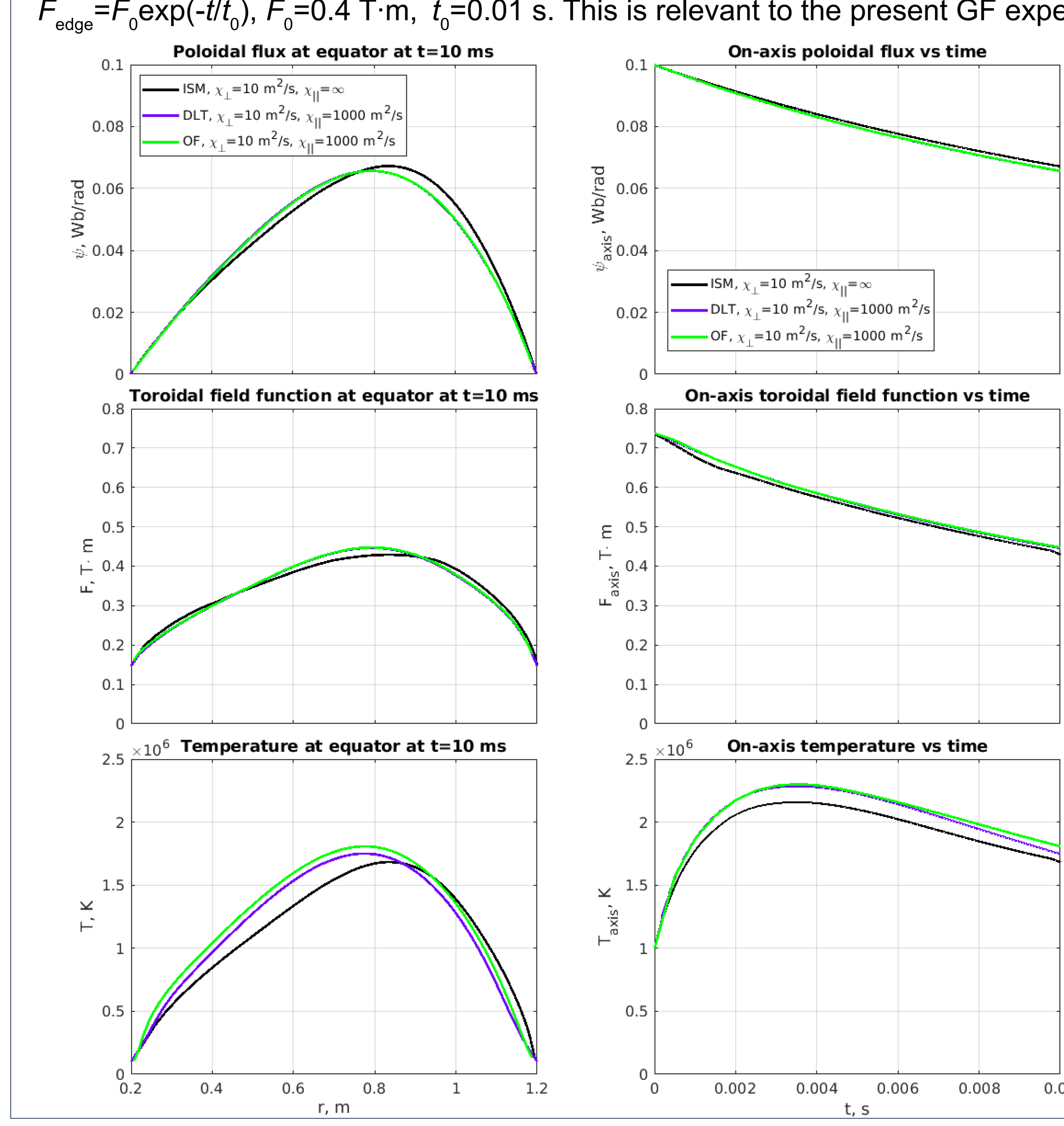
$$\chi_\perp = 10\ \text{m}^2\text{s}^{-1}$$





Test case 2: decaying shaft current

- The same as test case 1, but toroidal field at the boundary (shaft current) is decaying: $F_{edge}=F_0\exp(-t/t_0)$, F_0=0.4 T·m, t_0=0.01 s. This is relevant to the present GF experiment.



COMPRESSION GEOMETRY

- In this case we demonstrate capability of ISM-plasma to model compression geometry.
- Plasma is formed in a shaped domain and then compressed by moving right boundary; the boundary trajectory is given as a function of time and mimics the motion of liner.
- Plasma and boundaries are perfect conductors.

ISM-plasma numerical parameters:

$$N_r = 100,\quad N_z = 50,\quad N_C = 20,\quad dt = 2\cdot10^{-6}\ \text{s}$$

Initial plasma parameters:

$$\bar\psi = \frac{\psi}{\psi_{axis}},\quad \psi_{axis} = 0.14\ \text{Wb/rad}$$

$$F(\bar\psi) = F_{edge} + (F_{axis} - F_{edge})\bar\psi$$

$$F_{edge} = 0.2\ \text{T·m},\quad F_{axis} = 0.25\ \text{T·m}$$

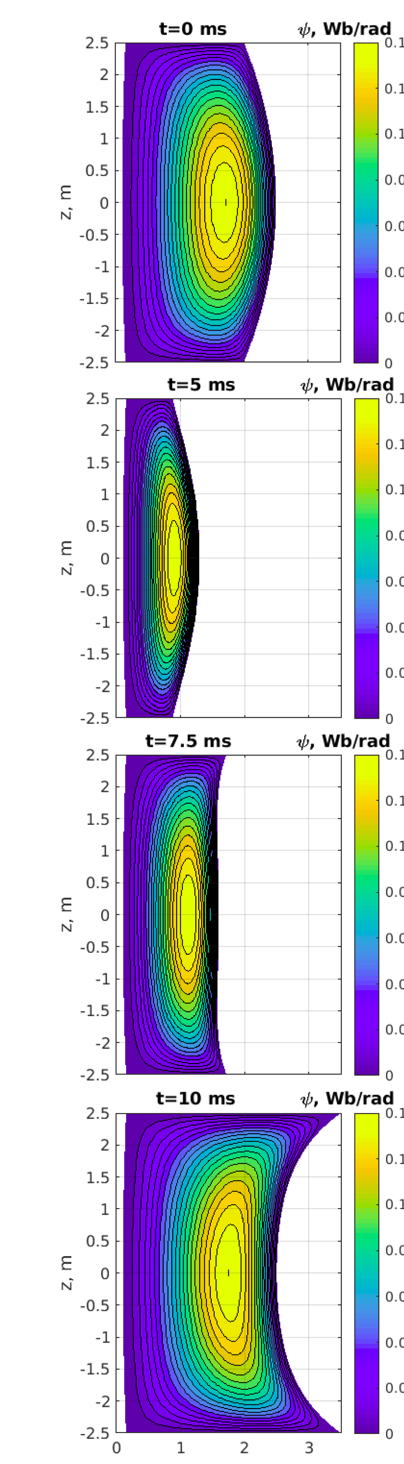
$$p(\bar\psi) = p_{axis}\bar\psi,\quad p_{axis} = 10^4\ \text{Pa}$$

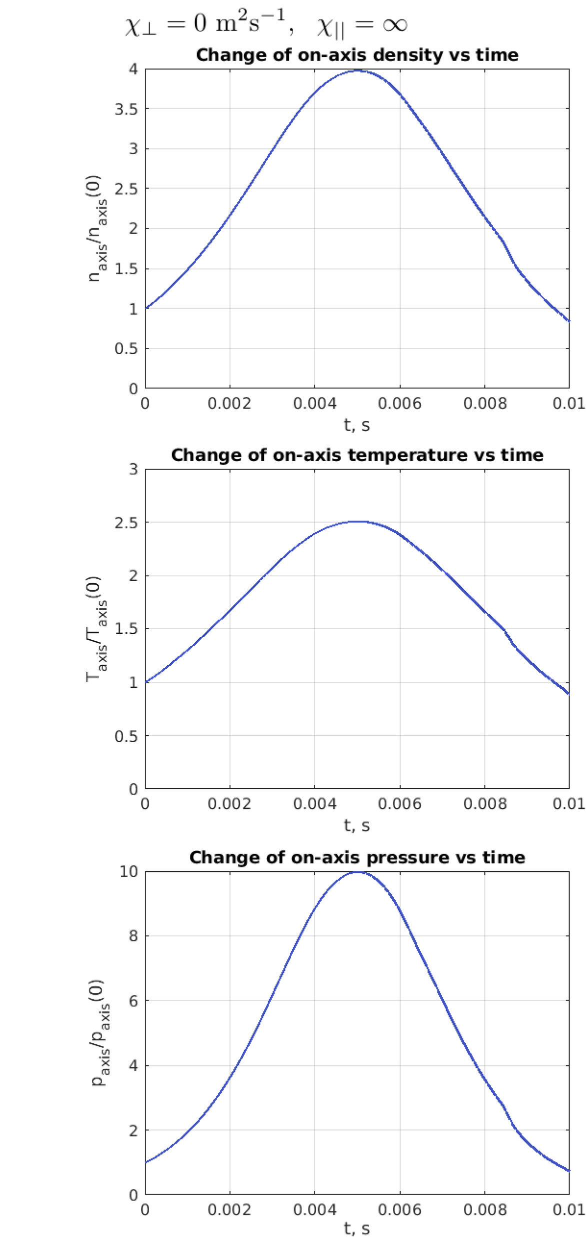
$$n(\bar\psi) = n_0\bar\psi,\quad n_0 = 7.24\cdot10^{20}\ \text{m}^{-3}$$

$$T(\bar\psi) = T_{axis}\bar\psi,\quad T_{axis} = 10^6\ \text{K}$$

$$\eta = 0\ \text{m}^2\text{s}^{-1}$$

$$\chi_\perp = 0\ \text{m}^2\text{s}^{-1},\quad \chi_\parallel = \infty$$





CONCLUSIONS

- The ISM-plasma code is developed in General Fusion for simulating resistive evolution of plasma equilibrium in settings appropriate to GF MTF systems.
- The code is based on a contour method: at every time step it alternates between solving the 2D Grad-Shafranov equilibrium on Eulerian mesh and advancing the 1D transport equations discretized on a set of Lagrangian ψ contours.
- The main advantages of the code are:
  - speed (time step is limited by slow resistive timescale);
  - capability to model plasma inside domain with moving boundaries (as in MTF compression systems).
- Considered test cases show a good comparison of ISM-plasma with other MHD codes. The code performs very well in a steady geometry under conditions relevant to the present GF Plasma Injector 3 (PI3) experiment.
- Future development of the ISM-plasma code includes:
  - addition of vacuum poloidal field in plasma domain;
  - addition of field diffusion into liner;
  - coupling of ISM-plasma with self-consistent model for the liner motion determined from the ISM-hydro code [4].

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